

$\sum_{p \leq n} \frac{1}{p} = \ln(\ln(n)) + O(1)$ : **An Exposition**  
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## 1 Introduction

It is well known that  $\sum_{p \leq n} \frac{1}{p} = \ln(\ln(n)) + O(1)$  where  $p$  goes over the primes. We give several known proofs of this.

We first present a proof that  $\sum_{p \leq n} \frac{1}{p} \geq \ln(\ln(n)) + O(1)$ . This is based on Euler's proof that  $\sum_p \frac{1}{p}$  diverges. We then present three proofs that  $\sum_{p \leq n} \frac{1}{p} \leq \ln(\ln(n)) + O(1)$ . The first one, essentially due to Mertens, does not use the prime number theorem. The second and third one do use the prime number theorem and hence are shorter.

For a complete treatment of Merten's proof that  $\sum_p \frac{1}{p}$  diverges, and how it compares with modern treatments, see the scholarly work of Villarino [4].

## 2 Euler's Proof that $\sum_{p \leq n} \frac{1}{p} \geq \ln(\ln(n)) + O(1)$

The proof here follows the one in [1].

**Lemma 2.1** For  $0 \leq x \leq 1/2$ ,  $-\ln(1-x) \leq x + x^2$ .

**Proof:**  $-\ln(1-x) = \int_0^x \frac{1}{1-t} dt$ . For  $0 \leq t \leq 1/2$ ,  $\frac{1}{1-t} \leq 1 + 2t$ . Hence

$$-\ln(1-x) = \int_0^x \frac{1}{1-t} dt \leq \int_0^x (1+2t) dt = x + x^2.$$

■

**Theorem 2.2**  $\sum_{p \leq n} \frac{1}{p} \geq \ln(\ln(n)) + O(1)$ .

**Proof:** Clearly

$$\sum_{j=1}^{\infty} \frac{1}{j} = (1 - \frac{1}{2} + \frac{1}{2^2} + \cdots)(1 - \frac{1}{3} + \frac{1}{3^2} + \cdots) \cdots = \frac{1}{1-2^{-1}} \times \frac{1}{1-3^{-1}} \times \cdots$$

which we rewrite as

$$\sum_{j=1}^{\infty} \frac{1}{j} = \prod_p (1 - p^{-1})^{-1}$$

We need a finite version of this statement. Let  $S_n$  be the set of natural numbers whose prime factors  $p$  are all  $\leq n$ . Then

$$\sum_{j \in S_n} \frac{1}{j} = \prod_{p \leq n} (1 - p^{-1})^{-1}.$$

Clearly  $\sum_{j \leq n} \frac{1}{j} \leq \sum_{j \in S_n} \frac{1}{j}$ . By integration  $\ln n \leq \sum_{j \leq n} \frac{1}{j}$ . Hence we have

$$\begin{aligned} \ln(n) &\leq \sum_{j \leq n} \frac{1}{j} \leq \sum_{j \in S_n} \frac{1}{j} = \prod_{p \leq n} (1 - p^{-1})^{-1} \\ \ln(\ln(n)) &\leq \sum_{p \leq n} -\ln(1 - p^{-1}). \end{aligned}$$

By Lemma 2.1

$$\sum_{p \leq n} -\ln(1 - p^{-1}) \leq \sum_{p \leq n} \frac{1}{p} + \frac{1}{p^2}.$$

Putting this all together we get

$$\sum_{p \leq n} \frac{1}{p} \geq \ln(\ln(n)) - \sum_{p \leq n} \frac{1}{p^2}$$

Since the second sum is bounded by  $\sum_{i=1}^{\infty} \frac{1}{i^2}$ , which converges, we have

$$\sum_{p \leq n} \frac{1}{p} \geq \ln(\ln(n)) - O(1).$$

■

**Note 2.3** If the above proof is done more carefully with attention paid to the constants you can obtain  $\sum_{p \leq n} \frac{1}{p} \geq \ln(\ln(n)) - 0.48$ . See [1].

### 3 Mertens Proof that Does Not Use the Prime Number Theorem

This is adapted from Landau's book [2]. He works a little harder and gets  $o(1)$  instead of  $O(1)$ .

We first need a weak form of the prime number theorem.

**Lemma 3.1**  $\pi(x) = O(x/\ln x)$ .

**Proof:** Let  $n$  be a positive integer. Clearly every prime  $p$  with  $n < p \leq 2n$  occurs in the prime factorization of the binomial coefficient  $\binom{2n}{n}$ . Therefore,

$$n^{\pi(2n) - \pi(n)} = \prod_{n < p \leq 2n} p \leq \prod_{n < p \leq 2n} p \leq \binom{2n}{n} \leq (1 + 1)^{2n} = 4^n.$$

Taking logs yields

$$\pi(2n) - \pi(n) \leq \frac{n \ln 4}{\ln n} \leq 2 \ln 4 \left( \frac{2n}{\ln(2n)} - \frac{n}{\ln n} \right)$$

for  $n \geq 8$ . If  $y \geq 16$  is a real number, let  $2n$  be the largest even integer with  $2n \leq y$ . Then  $\pi(y) - \pi(2n) \leq 1$  and  $|\pi(y/2) - \pi(n)| \leq 1$ . By increasing  $2 \ln 4$  to 4 we can absorb these errors and obtain

$$\pi(y) - \pi(y/2) \leq 4 \left( \frac{y}{\ln y} - \frac{y/2}{\ln(y/2)} \right)$$

for  $y \geq 16$ . Adding up this inequality for  $y = x, x/2, x/4, \dots$  yields

$$\pi(x) - \pi(16) \leq 4 \left( \frac{x}{\ln x} \right).$$

This yields the lemma. ■

We now need a result that is interesting in its own right.

**Proposition 3.2**

$$\sum_{p \leq x} \frac{\ln p}{p} = \ln x + O(1).$$

**Proof:** If  $n$  is a positive integer and  $p$  is a prime, the power of  $p$  dividing  $n!$  is

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots$$

Therefore,

$$\ln(n!) = \sum_{p \leq n} \ln p \left( \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots \right).$$

Changing  $\lfloor n/p \rfloor$  to  $n/p$  introduces an error of most 1, so we have

$$\sum_{p \leq n} \ln p \left\lfloor \frac{n}{p} \right\rfloor = n \sum_{p \leq n} \frac{\ln p}{p} + O(\sum_{p \leq n} \ln p).$$

Since there are  $\pi(n)$  terms in the sum, Lemma 1 implies that

$$O(\sum_{p \leq n} \ln p) = O(\pi(n) \ln n) = O(n).$$

Let's treat the higher terms:

$$\left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots < \frac{n}{p^2} (1 + p^{-1} + p^{-2} + \dots) = \frac{n}{p^2 - p}.$$

Therefore,

$$\sum_{p \leq n} \ln p \left( \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots \right) \leq n \sum_{p \leq n} \frac{\ln p}{p^2 - p} = O(n)$$

since  $\sum \ln p/(p^2 - p) \leq \sum_{j \geq 2} \ln j/(j^2 - j)$ , which converges.

Stirling's formula says that

$$\ln(n!) = n \ln n + O(n)$$

(this weak form can be proved by comparing  $\sum \ln j$  with  $\int \ln t \, dt$ ). Putting everything together yields

$$n \ln n + O(n) = n \sum_{p \leq n} \frac{\ln p}{p} + O(n).$$

Dividing by  $n$  yields the proposition for  $x = n$ . The error introduced by changing from  $x$  to  $n = \lfloor x \rfloor$  is absorbed by  $O(x)$ , so the proposition is proved. ■

The following lemma is well known. It is an analog of integration by parts for summations. It is easily proven by induction on  $n$ .

**Lemma 3.3** *Let both  $f_1, f_2, \dots$  and  $g_1, g_2, \dots$  be sequences of complex numbers. Then, for all  $m \leq n$ ,*

$$\sum_{i=m}^n f_i(g_{i+1} - g_i) = f_{n+1}g_{n+1} - f_m g_m - \sum_{i=m}^n g_{i+1}(f_{i+1} - f_i).$$

We can now prove the theorem.

**Theorem 3.4**  $\sum_{p \leq x} \frac{1}{p} = \ln \ln x + O(1)$ .

**Proof:** We have

$$f(x) = \sum_{p \leq x} \frac{\ln p}{p} = \ln x + r(x),$$

where  $r(x) = O(1)$ . Then

$$\begin{aligned} \sum_{p \leq x} \frac{1}{p} &= \sum_{p \leq x} \frac{\ln p}{p} \frac{1}{\ln p} = \sum_{n=2}^x \frac{f(n) - f(n-1)}{\ln n} \\ &= \sum_{n=2}^x \frac{\ln n - \ln(n-1)}{\ln n} + \sum_{n=2}^x \frac{r(n) - r(n-1)}{\log n}. \end{aligned}$$

Since

$$\ln n - \ln(n-1) = -\ln \left( 1 - \frac{1}{n} \right) = \frac{1}{n} + O(1/n^2),$$

and

$$\sum_{n=2}^x \frac{1}{n \ln n} = \ln \ln x + O(1),$$

we find that

$$\sum_{n=2}^x \frac{\ln n - \ln(n-1)}{\ln n} = \ln \ln x + O(1).$$

Summation by parts yields

$$\begin{aligned} \sum_{n=2}^x \frac{r(n) - r(n-1)}{\log n} &= \sum_{n=2}^x r(n) \left( \frac{1}{\ln n} - \frac{1}{\ln(n+1)} \right) + \frac{r(\lfloor x \rfloor)}{\ln(\lfloor x \rfloor + 1)} \\ &= O\left(\sum_{n=2}^x \frac{1/n}{(\ln n)^2}\right) + O(1) = O(1). \end{aligned}$$

Putting everything together yields the theorem. ■

## 4 A Proof that uses Summation by Parts

In this section we give the standard way to estimate  $\sum 1/p$  using the Prime Number Theorem.

**Theorem 4.1**  $\sum_{p \leq n} \frac{1}{p} = \ln(\ln(n)) + O(1).$

**Proof:** Let  $\pi(i)$  be the number of primes  $\leq i$ . Let  $g(i) = \pi(i-1)$  and  $f(i) = \frac{1}{i}$ . Let  $m = 2$ . Plugging these into Lemma 3.3 yields

$$\sum_{i=2}^n \frac{1}{i} (\pi(i) - \pi(i-1)) = \frac{1}{n+1} \pi(n) - \frac{1}{2} \pi(1) - \sum_{i=2}^n \pi(i) \left( \frac{1}{i+1} - \frac{1}{i} \right).$$

We need:

- $\pi(i) - \pi(i-1)$  is 1 if  $i$  is prime but 0 otherwise.
- $\pi(n) = \frac{n}{\ln n} + O\left(\frac{n}{\ln^2 n}\right)$  by the Prime Number Theorem (when it is proved with an error term).

We have

$$\pi(i) \left( \frac{1}{i+1} - \frac{1}{i} \right) = \frac{\pi(i)}{i(i+1)} = \frac{1}{(i+1) \ln i} + O\left(\frac{1}{(i+1) \ln^2 i}\right)$$

by the Prime Number Theorem. But this equals

$$\frac{1}{i \ln i} - \frac{1}{i(i+1) \ln i} + O\left(\frac{1}{(i+1) \ln^2 i}\right) = \frac{1}{i \ln i} + O\left(\frac{1}{(i+1) \ln^2 i}\right).$$

Therefore,

$$\sum_{p \leq n} \frac{1}{p} = \sum_{i=2}^n \frac{1}{i \ln i} + O\left(\frac{1}{(i+1) \ln^2 i}\right) = \ln(\ln(n)) + O(1),$$

where we have used

$$\sum_{i=2}^n \frac{1}{i \ln i} = \int_2^n \frac{1}{x \ln x} dx + O(1) = \ln(\ln(x)) + O(1)$$

and

$$\sum_{i=2}^n \frac{1}{(i+1) \ln^2 i} = O(1)$$

by the Integral Test. ■

## 5 A Proof that uses Integration by Parts

This is the same as the previous proof, with the summation by parts replaced by integration by parts in a Stieltjes integral.

**Theorem 5.1**  $\sum_{p \leq n} \frac{1}{p} = \ln(\ln(n)) + O(1)$ .

**Proof:** The preceding proof can be rewritten using Stieltjes integrals:

$$\sum_{p \leq x} \frac{1}{p} = \int_{1.9}^x \frac{1}{t} d\pi(t).$$

Integration by parts yields

$$\frac{\pi(x)}{x} + \int_{1.9}^x \frac{\pi(t)}{t^2} dt.$$

We use the Prime Number Theorem approximation  $\pi(x) = \frac{x}{\ln x} + O(\frac{x}{\ln^2 x})$  to obtain

$$\frac{1}{\ln x} + \int_{1.9}^x \frac{1}{t \ln t} + O\left(\int_{1.9}^x \frac{1}{t \ln^2 t}\right) = \ln(\ln(x)) + O(1).$$

■

## 6 What Else is Known

Rosser and Schoenfeld [3] have shown that, when  $n \geq 286$ ,

$$\ln(\ln n) - \frac{1}{2(\ln n)^2} + B \leq \sum_{p \leq n} \frac{1}{p} \leq \ln(\ln n) + \frac{1}{(2 \ln n)^2} + B,$$

where  $B = 0.261497212847643$ .

Even though the sum  $\sum_{p \leq n} \frac{1}{p}$  diverges, it grows very slowly:

- $\sum_{p \leq 10} \frac{1}{p} = 1.176$
- $\sum_{p \leq 10^6} \frac{1}{p} = 2.887$
- $\sum_{p \leq 10^9} \frac{1}{p} = 3.293$
- $\sum_{p \leq 10^{100}} \frac{1}{p} \sim 5.7$

## References

- [1] J. Kraft and L. Washington. *An introduction to Number Theory and Cryptography*. CRC Press, 2014.
- [2] E. Landau. *Handbuch der Lehre von der Verteilung der Primzahlen*. Chelsea Publishing Co, 1953.
- [3] J. Roser and L. Schoenfeld. Approximate formulas for some properties of prime numbers. *Illinois Journal of Mathematics*, pages 64–94, 1962.
- [4] M. B. Villarino. Merten’s proof of Mertens’ theorem, 2005. <http://arxiv.org/abs/1205.3813>.